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Technical Notes

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Simple High-Order Bounded Convection Scheme to Model Discontinuities

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Introduction

THE correct modeling of convection, without introducing excessive artificial dissipation while retaining high-order accuracy, stability, boundedness, and algorithmic simplicity, plays a key role in reproducing complex flow physics, such as sharp gradients of transported flow variables often encountered in turbulent flows or compressible flows in the presence of shock. To obtain stable and bounded solutions, a first-order upwind-difference scheme is often adopted; however, this scheme (with a leading second-order truncation error resembling a viscous process) is highly diffusive in situations in which the flow direction is oblique or skewed relative to the numerical grid. To remedy this, a high-order upwind scheme is usually preferred because of its lack of numerical dissipation; however, it also inevitably generates oscillations when excited by features like a shock or steep variations of the transported properties.

The high-order upwind convection scheme QUICK,[1] for example, although handling convective transport for problems with smoothly varying dependent variables, introduces unrealistic overshoots and oscillatory results in regions where the convected variables experience sharp changes in gradients or discontinuities under highly convective conditions. To overcome this, a number of schemes (e.g., SHARP[2] and SMART[3]) have been proposed; however, these schemes, although performing well in passive scalar problems, are not bounded in situations like shock tube flows.[4]

Total variation diminishing (TVD) schemes[5–8] possess several attractive features, such as delivering well-resolved, nonoscillatory discontinuities and convergent solutions. It is possible to derive methods with a high order of accuracy that are TVD. In the present study, a new high-order-accurate convection scheme that satisfies the TVD principle and is based on nonlinear functional relations between normalized variables is presented. The new scheme is to be applied to a pure convection problem and compressible flow with a discontinuity of the transported variable.

Mathematical and Numerical Approach

The variation of a convected variable within a control volume can be represented by a curve linking nodes ϕ_D, ϕ_C, ϕ_U, which represent downstream, central, and upstream nodes, respectively, with respect to the convecting velocity. If the functional relationship between the nodes is prescribed, the facial value that is needed in the control volume formulation can be determined. To facilitate the discussion and further developments, the original variables are transformed into normalized variables,[2] defined as

$$\tilde{\phi} = \frac{\phi - \phi_U}{\phi_D - \phi_U} \tag{1}$$

The advantage of the transformed normalized variable is that the face value $\tilde{\phi}_f$ depends on $\tilde{\phi}_C$ only, because $\tilde{\phi}_D = 1$ and $\tilde{\phi}_U = 0$.

Before proceeding to the derivation of the new scheme, it is essential to discuss the TVD constraints. Consider a scalar equation,

$$\frac{\partial \phi}{\partial t} + a\frac{\partial \phi}{\partial x} = 0 \tag{2}$$

To satisfy the TVD principle, the following constraints have to be satisfied[6,7]:

$$0 \leq \frac{\Psi(r_f)}{r_f} \leq 2, \qquad 0 \leq \Psi(r_f) \leq 2 \tag{3}$$

where $\Psi(r_f)$ is the flux limiter and $r_f = \tilde{\phi}_C/(1 - \tilde{\phi}_C)$. From these relations, the functional relationship between $\tilde{\phi}_f$ and $\tilde{\phi}_C$, which satisfies the TVD principle, can be expressed as

$$\begin{aligned} &\tilde{\phi}_f \leq 1, \qquad \tilde{\phi}_f \leq 2\tilde{\phi}_C \\ &\tilde{\phi}_f \geq \tilde{\phi}_C \qquad 0 < \tilde{\phi}_C < 1 \\ &\tilde{\phi}_f = \tilde{\phi}_C \qquad \tilde{\phi}_C \leq 0 \quad \text{or} \quad \tilde{\phi}_C \geq 1 \end{aligned} \tag{4}$$

Therefore, to satisfy the TVD principle, the facial value $\tilde{\phi}_f$ must lie within the dashed-line area in the monotonic region and on the line $\tilde{\phi}_f = \tilde{\phi}_C$ outside the monotonic region (Fig. 1).

Several functional relations between the normalized convected face value $\tilde{\phi}_f$ and the normalized adjacent upstream node value $\tilde{\phi}_C$ for different schemes are shown in Tables 1 and 2. Most of the functional relations are linear, apart from SMART and SHARP, which are nonlinear in nature. Figure 1 shows four popular schemes: upwind, QUICK, SMART, and SHARP, plotted on the normalized variable diagram (NVD). It is clear that, apart from the upwind scheme, the rest of the three schemes do not satisfy the TVD constraints. Among these, QUICK and SHARP have a higher proportion of relations located outside the TVD range and it is not surprising to speculate

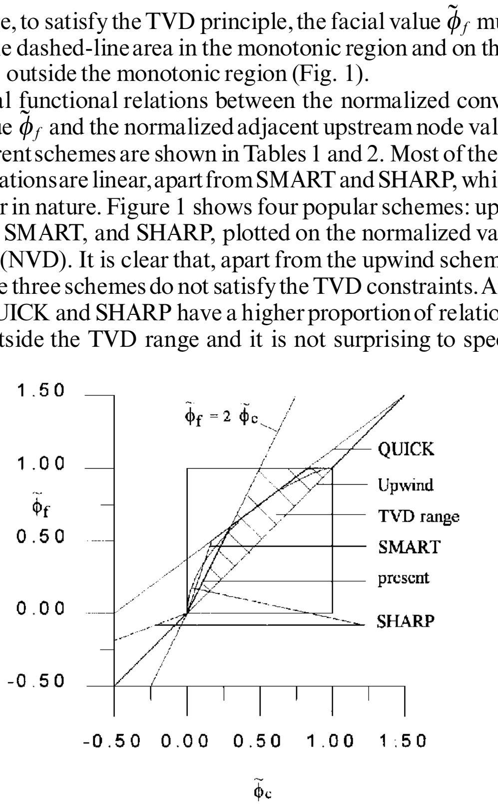

Fig. 1 Normalized variable diagram of schemes.

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Table 1 Functional relations between normalized variables of different schemes

Scheme	Functional relations
First-order upwind	$\tilde{\phi}_f = \tilde{\phi}_C$
Second-order upwind	$\tilde{\phi}_f = 3/2\tilde{\phi}_C$
Third-order QUICK	$\tilde{\phi}_f = 3/4\tilde{\phi}_C + 3/8$
Third-order SMART	$\tilde{\phi}_f = \tilde{\phi}_C$ if $\tilde{\phi}_C \ni [0, 1]$ $\tilde{\phi}_f = 3\tilde{\phi}_C$ if $\tilde{\phi}_C \in [0, 1/6]$ $\tilde{\phi}_f = 3/4\tilde{\phi}_C + 3/8$ if $\tilde{\phi}_C \in [1/6, 5/6]$ $\tilde{\phi}_f = 1$ if $\tilde{\phi}_C \in [5/6, 1]$
SHARP	$\tilde{\phi}_f = ([\tilde{\phi}_C(1 - \tilde{\phi}_C)^3]^{1/2} - \tilde{\phi}_C^2)/(1 - 2\tilde{\phi}_C)$ if $\tilde{\phi}_C \in [0, 0.35]$ or $\tilde{\phi}_C \in [0.65, 1]$ $\tilde{\phi}_f = 3/4\tilde{\phi}_C + 3/8$ if $\tilde{\phi}_C \in [0.35, 0.65]$ or $\in [-\infty -1]$ or $\in [1.5, \infty]$ $\tilde{\phi}_f = 3/8\tilde{\phi}_C$ if $\tilde{\phi}_C \in [-1, 0]$ $\tilde{\phi}_f = \tilde{\phi}_C$ if $\tilde{\phi}_C \in [1, 1.5]$

Table 2 Functional relations between normalized variables of the present scheme

Functional relations
$\tilde{\phi}_f = \tilde{\phi}_c$ if $\tilde{\phi}_c \ni [0, 1]$
$\tilde{\phi}_f = 2\tilde{\phi}_c$ if $\tilde{\phi}_c \in [0, 3/10]$
$\tilde{\phi}_f = 3/4\tilde{\phi}_c + 3/8$ if $\tilde{\phi}_c \in [3/10, 5/6]$
$\tilde{\phi}_f = 1$ if $\tilde{\phi}_c \in [5/6, 1]$

that the schemes might have the tendency to produce overshoots and oscillations, in regions of sharp gradients, like a shock. To remedy this and still maintain a high order of accuracy, a new scheme is proposed, and this is shown in Fig. 1. It is clearly seen that the present scheme lies within the TVD range. This scheme possesses a nonlinear NVD characteristic, which passes through the point (0, 0), (1, 1), and (0.5, 0.75), with a slope of 0.75 at (0.5, 0.75), as recommended by Leonard.[2] Because the regions for the present scheme coinciding with the QUICK scheme are slightly less than those of the SMART scheme, the accuracy of the present scheme might be slightly lower. Therefore, slightly more diffusive results might be expected.

Results and Discussion

Two-Dimensional Pure Convection of a Boxed-Shape Profile

The first case considered is a steady inviscid scalar convection problem. The transported scalar is convected at a constant velocity, which is skewed at an angle, $\theta = 45$ deg, relative to the numerical grid, with a box-shape step change of the transported scalar at the inflow boundary. The length of the computational domain is one in both the X and Y directions and the mesh size adopted was 41×41. Four schemes, namely, upwind, QUICK, SMART, and the newly proposed scheme, were used. The predicted results of various schemes are compared with the exact solution shown in Fig. 2. Although the QUICK scheme generates over- and undershoots and the upwind scheme indicates an excessive level of numerical diffusion, both the present scheme and SMART remain bounded and achieve higher-order accuracy.

Unsteady One-Dimensional Shock Tube Problem

In contrast to the previous steady linear problem, attention is focused here on an unsteady nonlinear shock-tube problem. The governing equations for this flow were detailed by Yee et al.[8] In this application, a forward time integration scheme was adopted. The initial conditions of the problem consist of two stagnant regions, $0 \leq x \leq 1$ with different densities and pressures separated by a diaphragm placed at $x = 0.5$. At time $t = 0$, the diaphragm is ruptured and a series of compression waves rapidly coalesce into a normal shock. Simulations with $\Delta x = 0.01$ and $\Delta t = 0.02$

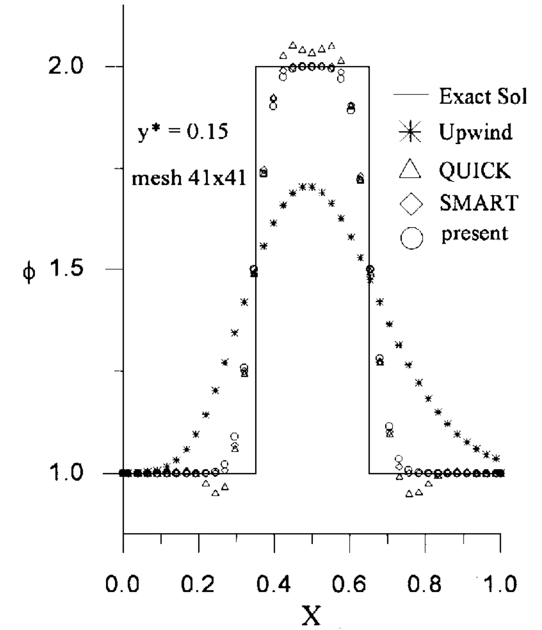

Fig. 2 Pure convection of a boxed-shape profile.

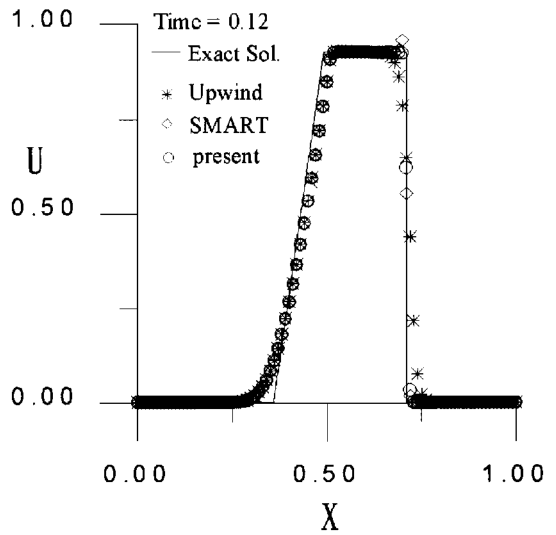

Fig. 3 Predicted velocity distributions of shock tube flow.

considered here focus on the case before any wave has reached the boundaries.

The superior performance of the present scheme can be ascertained by observing Fig. 3, showing the velocity variations at time $t = 0.12$. The diffusive nature of the upwind scheme can be identified. It can also be clearly seen that the SMART scheme, which maintains bounded solutions in the previous case, produces high levels of oscillation, although the scheme still preserves high-order accuracy. In sharp contrast, the present scheme remains bounded and exhibits high-order accuracy.

Conclusions

A new high-order accurate convection scheme, which satisfies the TVD principle and is based on nonlinear functional relations between the normalized variables, has been developed. Applications of the scheme, together with QUICK and SMART, to steady linear two-dimensional passive scalar problems indicate that both the proposed scheme and SMART returned the most accurate and least oscillatory results; by contrast, overshoots and undershoots were found in the QUICK solutions. The new scheme was applied to one-dimensional compressible shock tube flows. Predicted results, contrasted with exact solutions and the SMART scheme, indicate that the present method returned the most accurate and least oscillatory results.

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Thin vs Full Navier–Stokes Computation for High-Angle-of-Attack Aerodynamics

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Introduction

THE forebody of a flight vehicle can be typified by a slender body of revolution. Because such an axisymmetric body is numerically and experimentally tractable, it has been the object of much research over the years. A number of experimenters (e.g., Refs. 1–3) have noted that for such bodies, there is an angle-of-attack range (roughly, from 30 to 65 deg) for which minute imperfections at the tip (e.g., dust accumulation, surface roughness) can cause large asymmetries in the flow pattern. They found that as the angle of attack increases from 30 to 65 deg, the form of the side-force response as a function of the roll angle changes from a continuous, almost periodic variation to a virtually discontinuous square wave, commonly called a bistable variation. For angles of attack beyond 65 deg, the crossflow past the cylindrical part of the body becomes virtually identical to the flow past a two-dimensional cylinder: Vortex shedding occurs, and as the angle of attack tends toward 90 deg the mean side force tends toward zero.

That minute perturbations of body shape can result in finite asymmetries suggests the existence of inherent instabilities in the expected symmetric flow. This is certainly the case as the angle of attack approaches 90 deg since the observed Kármán vortex street beyond the body is well known to be caused by an absolute instability of the symmetric flow: one which would remain even after removal of any perturbation that initiated it. However, in numerical predictions for angles of attack between 30 and 65 deg both for laminar[4–6] and for turbulent[7] flows, the existence of convective instability of the symmetric flow was found. More recent evidence to support the existence of convective instability has been obtained from several experiments.[8,9]

Numerical methods that can simulate the asymmetry phenomenon must account for the fluid viscosity. The degree to which the viscous terms are considered usually affects the complexity and execution time requirements of the computer models. In previous work, we have employed the thin-layer Navier–Stokes (TLNS) model. This model considers only the effect of the viscosity on the flow normal to the body. Objections have been raised regarding the justification for utilizing the TLNS model to study flow asymmetries, because the viscosity effect of the circumferential and streamwise flow components might become important at the high attack angles of interest.

The objective of this work is to test this question by comparing the three-dimensional Beam–Warming thin-layer Navier–Stokes (BW TLNS) algorithm and Beam–Warming full Navier–Stokes (BW FNS) algorithm where all viscous terms are kept. The three-dimensional flux-splitting thin-layer Navier–Stokes (FS TLNS) algorithm will be used as an additional reference.

Numerical Algorithms

The conservation equations of mass, momentum, and energy can be represented in a flux-vector form that is convenient for numerical simulation as

$$\partial_\tau \hat{Q} + \partial_\xi(\hat{F} + \hat{F}_v) + \partial_\eta(\hat{G} + \hat{G}_v) + \partial_\zeta(\hat{H} + \hat{H}_v) = 0 \quad (1)$$

For body-conforming coordinates and attached or mildly separated high-Reynolds number flow, if ζ is the coordinate leading away from the surface, the thin-layer approximation can be applied, which yields[10,11]

$$\partial_\tau \hat{Q} + \partial_\xi \hat{F} + \partial_\eta \hat{G} + \partial_\zeta \hat{H} = Re^{-1}\partial_\zeta \hat{S}^\zeta \quad (2)$$

where only viscous terms in ζ are retained. These have been collected into the vector $\hat{S}^\zeta$, and the nondimensional Reynolds number Re is factored from the viscous flux term. It is Eq. (2) that is solved in the TLNS model.

Flux-Splitting Algorithm

The flux-splitting algorithm[12] may be used to solve Eq. (2). The algorithm is an implicit scheme and uses flux-vector splitting and upwind spatial differencing for the convection terms in one coordinate direction (nominally streamwise). By using upwind differencing for the convective terms in the streamwise direction and central differencing in the other directions, a two-factor implicit approximately factored algorithm is obtained, which is unconditionally stable for a representative model wave equation.

Beam–Warming Algorithms

The FS TLNS algorithm cannot be extended to an FNS solver due to the flux-splitting technique limitation, which does not allow (implicit) consideration of the viscous terms in the streamwise direction. The Beam–Warming algorithm[13] uses central differences in all three directions (ξ, η, ζ) and, therefore, allows the extension of the code to an FNS solver. Equation (1) can be written as

$$\partial_\tau \hat{Q} + \partial_\xi \hat{F} + \partial_\eta \hat{G} + \partial_\zeta \hat{H} = Re^{-1}\big[\partial_\xi(\hat{P}^\xi + \hat{P}^\eta + \hat{P}^\zeta) + \partial_\eta(\hat{R}^\xi + \hat{R}^\eta + \hat{R}^\zeta) + \partial_\zeta(\hat{S}^\xi + \hat{S}^\eta + \hat{S}^\zeta)\big] \quad (3)$$

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